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Note

The Dual of the Ahlswede–Zhang Identity

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Let Ω be the set $\{1, 2, \ldots, n\}$, and 2^n be the set of all subsets of Ω. Let $\mathscr{F}$ be the set of all $\mathscr{A} \subseteq 2^n$ with $\mathscr{A} \neq \varnothing$ and $\varnothing \notin \mathscr{A}$, where $\varnothing$ is the empty set. Similarly let $\mathscr{G}$ be the set of all $\mathscr{B} \subseteq 2^n$ with $\mathscr{B} \neq \varnothing$ and $\Omega \notin \mathscr{B}$. For $\mathscr{A} \in \mathscr{F}$ and $X \subseteq \Omega$ put

$$Z_{\mathscr{A}}(X) = \begin{cases} \varnothing & \text{if there is no } A \in \mathscr{A} \text{ with } A \subseteq X, \\ \bigcap\{A \in \mathscr{A}, A \subseteq X\} A & \text{otherwise.} \end{cases}$$

Also for $\mathscr{B} \in \mathscr{G}$ and $X \subseteq \Omega$ put

$$Z^*_{\mathscr{B}}(X) = \begin{cases} \Omega & \text{if there is no } B \in \mathscr{B} \text{ with } X \subseteq B, \\ \bigcup\{B \in \mathscr{B}, X \subseteq B\} B & \text{otherwise.} \end{cases}$$

The Ahlswede–Zhang identity [1] is

$$\Delta = 1 \text{ where } \Delta = \sum |Z_{\mathscr{A}}(X)| \Big/ \left(|X| \binom{n}{|X|} \right), \tag{1}$$

and summation is over all $X \subseteq \Omega$ with $X \neq \varnothing$. The dual of (1) which we

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here present is

$$\Gamma = \Phi \text{ where } \Gamma = \sum |Z^*_{\mathscr{B}}(X)| \Big/ \left((n - |X|) \binom{n}{|X|} \right), \tag{2}$$

and summation is over all $X \subseteq \Omega$ with $X \neq \Omega$, and

$$\Phi = n\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} \right).$$

Note that (1) is only concerned with the antichain of minimal elements of $\mathscr{A}$. On the other hand (2) is only concerned with the antichain of maximal elements of $\mathscr{B}$. Every maximal chain of subsets of Ω has an X with $Z_{\mathscr{A}}(X) \neq \varnothing$, and a Y with $Z^*_{\mathscr{B}}(Y) \neq \Omega$. Interestingly, the least common multiple l.c.m. of the denominators in (1) or (2) is the l.c.m. of $1, 2, \ldots, n$.

To prove that (1) *and* (2) *are equivalent*. Let us write c for complement. We may assume $A \in \mathscr{A}$ iff $cA \in \mathscr{B}$. Let $X \subseteq \Omega$ be given with $X \neq \varnothing$. Then there is an $A \in \mathscr{A}$ with $A \subseteq X$ iff there is $cA \in \mathscr{B}$ with $cX \subseteq cA$. When there is an $A \in \mathscr{A}$ with $A \subseteq X$ then

$$\begin{aligned} c\left(\bigcap\{A \in \mathscr{A}, A \subseteq X\}A\right) &= \bigcup\{A \in \mathscr{A}, cX \subseteq cA\}cA \\ &= \bigcup\{B \in \mathscr{B}, cX \subseteq B\}B, \end{aligned}$$

so

$$c(Z_{\mathscr{A}}(X)) = Z^*_{\mathscr{B}}(cX). \tag{3}$$

Now (3) also holds when there is no $A \in \mathscr{A}$ with $A \subseteq X$, because then $Z_{\mathscr{A}}(X) = \varnothing$ and $Z^*_{\mathscr{B}}(cX) = \Omega$. Next we use (3) to get

$$\begin{aligned} \Gamma &= \sum\{\varnothing \neq Y \subseteq \Omega\} |Z^*_{\mathscr{B}}(cY)| / (n - |cY|) \binom{n}{|cY|} \\ &= \sum\{\varnothing \neq Y \subseteq \Omega\} (n - |Z_{\mathscr{A}}(Y)|) / |Y| \binom{n}{|Y|} \\ &= 1 + \Phi - \Delta, \end{aligned}$$

so (1) and (2) are equivalent.

We discovered (2) when considering the case of (1) with $A \in \mathscr{A}$ iff $cA \in \mathscr{A}$. Now we give a direct proof of (2) in the style of [2].

LEMMA 1. *Let* $1 \leq b \leq n$. *If* $\lambda(r) = (n - b)\binom{b}{r} / (n - r)\binom{n}{r}$ *then* $\Lambda = b/n$ *where* $\Lambda = \lambda(b) + \lambda(b - 1) + \cdots + \lambda(1)$.

Proof. Using the identity $\binom{m}{s} = (m - s + 1)\binom{m}{s-1}/s$ we get

$$\left\{\binom{b}{r}\Big/\binom{n}{r}\right\} + \lambda(r-1) = \binom{b}{r-1}\Big/\binom{n}{r-1}. \tag{4}$$

The lemma follows by using (4) with $r = b, b-1, \ldots, 2$ from the left of Λ.

LEMMA 2[1]. *Let $\mathscr{D}, \mathscr{E} \in \mathscr{G}$ and put $\mathscr{D} \wedge \mathscr{E} = \{D \cap E: D \in \mathscr{D}, E \in \mathscr{E}\}$. Then $\mathscr{D} \wedge \mathscr{E}$, $\mathscr{D} \cup \mathscr{E} \in \mathscr{G}$ and for all $\Omega \neq X \subseteq \Omega$ we have*

$$|Z^*_{\mathscr{D}\cup\mathscr{E}}(X)| = |Z^*_{\mathscr{D}}(X)| + |Z^*_{\mathscr{E}}(X)| - |Z^*_{\mathscr{D}\wedge\mathscr{E}}(X)|. \tag{5}$$

Proof.

Case 1. There is no $D \in \mathscr{D}$ with $X \subseteq D$ and no $E \in \mathscr{E}$ with $X \subseteq E$. Here each $|Z^*|$ in (5) is n, from a Z^* which is Ω.

Case 2. There is no $D \in \mathscr{D}$ with $X \subseteq D$ but there is an $E \in \mathscr{E}$ with $X \subseteq E$. Here (5) holds because $Z^*_{\mathscr{D}}(X) = Z^*_{\mathscr{D}\wedge\mathscr{E}}(X) = \Omega$ and

$$Z^*_{\mathscr{D}\cup\mathscr{E}}(X) = \bigcup\{X \subseteq B \in \mathscr{D} \cup \mathscr{E}\}B = \bigcup\{X \subseteq B \in \mathscr{E}\}B = Z^*_{\mathscr{E}}(X).$$

Of course we can exchange $\mathscr{D}$ and $\mathscr{E}$ in this Case 2.

Case 3. There is a $D \in \mathscr{D}$ with $X \subseteq D$ and an $E \in \mathscr{E}$ with $X \subseteq E$. Then

$$\begin{aligned} Z^*_{\mathscr{D}\cup\mathscr{E}}(X) &= \left(\bigcup\{X \subseteq D \in \mathscr{D}\}D\right) \cup \left(\bigcup\{X \subseteq E \in \mathscr{E}\}E\right) \\ &= Z^*_{\mathscr{D}}(X) \cup Z^*_{\mathscr{E}}(X), \qquad \text{and} \\ Z^*_{\mathscr{D}\wedge\mathscr{E}}(X) &= \cup\{X \subseteq D \in \mathscr{D}, X \subseteq E \in \mathscr{E}\}(D \cap E) = Z^*_{\mathscr{D}}(X) \cap Z^*_{\mathscr{E}}(X), \end{aligned}$$

so (5) holds and the lemma is proved.

Induction Proof of (2).

Case 1. $|\mathscr{B}| = 1$ so $\mathscr{B} = \{B\}$ and $B = \varnothing$. Then $Z^*(\varnothing) = \varnothing$ but $Z^* = \Omega$ otherwise, and (2) follows easily.

Case 2. $|\mathscr{B}| = 1$ so $\mathscr{B} = \{B\}$ and $B \neq \varnothing$. Then $Z^*(X) = B$ if $X \subseteq B$ but $Z^* = \Omega$ otherwise. For $X \neq \varnothing$ we put $Z^* = Z_1^* - Z_2^*$, where Z_1^* is the Z^* of Case 1 for which (2) holds. That the sum of the other terms of Γ is zero follows immediately from Lemma 1.

[1]T. D. Thu apologises for not giving the details for (3) of [2] corresponding to this Lemma 2.

Case 3. We assume (2) holds for $1 \leq |\mathscr{B}| < h$ and consider the case $\mathscr{B} = \{B_1, B_2, \ldots, B_h\}$. Let $\mathscr{D} = \{B_1, B_2, \ldots, B_{h-1}\}$ and $\mathscr{E} = \{B_h\}$. Then for Lemma 2 we have $|\mathscr{D}|, |\mathscr{E}|, |\mathscr{D} \wedge \mathscr{E}| < h$. So if we sum (2) over each of the $|Z^*|$ in the right hand side of (5), by our induction hypothesis we get $\Phi + \Phi - \Phi$, which is the right hand side of (2). Hence (2) holds for $\mathscr{D} \cup \mathscr{E} = \mathscr{B}$.

Algorithm to Evaluate Z. Let $\mathscr{A} \in \mathscr{F}$ be given. We iteratively define $\alpha(X)$ and $Z(X)$ for $X \subseteq \Omega$. Whenever $\alpha(X) = 0$, $Z(X) = \varnothing$. Let $\delta(X) = \{Y: Y \subseteq X, |Y| = |X| - 1\}$. Put $\alpha(\varnothing) = 0$. Suppose α and Z are known for $|X| < h$. Then for $|X| = h$ do:

Case 1. There is a $Y \in \delta(X)$ with $\alpha(Y) = 1$. Put $\alpha(X) = 1$ and $Z(X) = \bigcap\{Y \in \delta(X), \alpha(Y) = 1\}Z(Y)$.

Case 2. Not Case 1. If $X \in \mathscr{A}$ then $\alpha(X) = 1$ and $Z(X) = X$. If $X \notin \mathscr{A}$ then $\alpha(X) = 0$.

Of course there is a corresponding algorithm for Z^*.

References

1. R. Ahlswede and Z. Zhang, An identity in combinatorial extremal theory, *Adv. Math.* **80** (1990), 137–151.
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